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Dietzenbacher, Bas; Borm, Peter; Hendrickx, Ruud

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**THE PROCEDURAL EGALITARIAN SOLUTION**

By

Bas Dietzenbacher, Peter Borm, Ruud Hendrickx

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# The Procedural Egalitarian Solution

Bas Dietzenbacher<sup>\*†</sup>      Peter Borm<sup>\*</sup>      Ruud Hendrickx<sup>\*</sup>

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## Abstract

In this paper we introduce and analyze the procedural egalitarian solution for transferable utility games. This new concept is based on the result of a coalitional bargaining procedure in which egalitarian considerations play a central role. The procedural egalitarian solution is the first single-valued solution which coincides with the constrained egalitarian solution of Dutta and Ray (1989) on the class of convex games and which exists for *any* TU-game.

**Keywords:** egalitarianism, egalitarian procedure, procedural egalitarian solution, egalitarian stability, constrained equal awards rule

**JEL classification:** C71

## 1 Introduction

Egalitarianism is a paradigm of economic thought that favors the idea of equality. Economic equality, or equity, refers to the concept of fairness in economics and underlies many theories of distributive justice. Starting from the seminal work of Rawls (1971) in which equality plays a central role in two fundamental principles of justice, egalitarianism and equity have inspired scientists within several areas, e.g. social philosophy and welfare economics. Young (1995) provides a rich survey on equity concepts in both theoretical and practical contexts. We focus on the role of egalitarianism in distributive justice applied to coalitional arrangements which affect the distribution of joint revenues among cooperating participants.

Dutta and Ray (1989) introduced a concept of egalitarianism under participation constraints for transferable utility games. A transferable utility game describes an allocation problem for a set of cooperating players in which the economic possibilities of all subcoalitions are taken into account. The *constrained egalitarian solution* of Dutta and Ray (1989) uses a specific Lorenz criterion to select a payoff allocation. Their most important result states that the constrained egalitarian solution selects at most one feasible allocation, despite the partial ordering generated by the Lorenz criterion. However, existence of the constrained egalitarian solution is only guaranteed for the special class of convex games.

The constrained egalitarian solution is well-studied on the class of convex games. Dutta and Ray (1989) showed that the constrained egalitarian solution of a convex game cannot be blocked by any subcoalition, i.e. it is an element of the core. Dutta (1990) axiomatically characterized the constrained egalitarian solution on the class of convex games using consistency properties for reduced games of Davis and Maschler (1965) and Hart and Mas-Colell

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<sup>\*</sup>CentER and Department of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands

<sup>†</sup>E-mail: b.j.dietzenbacher@tilburguniversity.edu

(1989). Other characterizations of the constrained egalitarian solution on the class of convex games are provided by Klijn, Slikker, Tijs, and Zarzuelo (2000) and Arin, Kuipers, and Vermeulen (2003).

Another line of research studies egalitarian concepts similar to the constrained egalitarian solution for a wider class of transferable utility games. Branzei, Dimitrov, and Tijs (2006) extended the computational algorithm for locating the constrained egalitarian solution of convex games to superadditive games by introducing the equal split-off set. Arin and Iñarra (2001) applied an egalitarian criterion to the core of balanced games by introducing the egalitarian core which satisfies the consistency property for reduced games of Davis and Maschler (1965). Both the equal split-off set and the egalitarian core coincide with the constrained egalitarian solution on the class of convex games. The most important shortcoming of these notions is that they generally lack the fundamental uniqueness property of the constrained egalitarian solution. To our knowledge, no appropriate egalitarian, single-valued solution concept is defined in the literature which coincides with the constrained egalitarian solution on the class of convex games and exists for any TU-game.

In this paper we introduce the *procedural egalitarian solution* as an egalitarian solution concept for which existence and uniqueness is guaranteed for *any* transferable utility game. Moreover, it coincides with the constrained egalitarian solution of Dutta and Ray (1989) on the class of convex games. The procedural egalitarian solution follows from an egalitarian procedure which is inspired by ideas underlying the average rules for cooperative TU-games of Sugumaran, Thangaraj, and Ravindran (2013). The egalitarian procedure models a natural way of negotiating by members of coalitions about an egalitarian distribution of their worth, taking into account their coalitional egalitarian externalities. The egalitarian procedure converges to a steady state in which each player has acquired a claim attainable in one or more egalitarian admissible coalitions. Using the constrained equal awards rule, the procedural egalitarian solution allocates the worth of the grand coalition in an egalitarian way among the players, taking into account their claims.

Selten (1972) proved that egalitarian allocations successfully explain outcomes of experimental cooperative games. Experimental evidence clearly suggests that equity considerations have a strong influence on observed payoff divisions. Coalition members look for easily accessible cues like equitable shares in order to form aspiration levels for their payoffs (cf. Selten (1987)). The egalitarian procedure seamlessly connects this phenomenon with transferable utility games.

This paper is organized in the following way. Section 2 provides an overview of the basic game theoretic notions and notations. Section 3 formally introduces the egalitarian procedure and studies its underlying structure. In Section 4, we introduce the procedural egalitarian solution, we derive some of its properties and we show that it coincides with the constrained egalitarian solution on the class of convex games. Section 5 concludes and formulates some suggestions for future research.

## 2 Preliminaries

Let  $N$  be a nonempty and finite set of *players*. The set of all *coalitions* is denoted by  $2^N = \{S \mid S \subseteq N\}$ . For any  $i, j \in N$  the involution  $\sigma^{i,j} : 2^N \rightarrow 2^N$  is for all  $S \in 2^N$  given by

$$\sigma^{i,j}(S) = \begin{cases} (S \cup \{j\}) \setminus \{i\} & \text{if } i \in S \text{ and } j \notin S; \\ (S \cup \{i\}) \setminus \{j\} & \text{if } i \notin S \text{ and } j \in S; \\ S & \text{if } i, j \in S \text{ or } i, j \notin S. \end{cases}$$

A collection of coalitions  $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$  is called a *cover* if  $\bigcup_{S \in \mathcal{B}} S = N$ . A set  $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$  is called *balanced* if there exists a function  $\lambda : 2^N \setminus \{\emptyset\} \rightarrow [0, 1]$  with  $\sum_{S \in 2^N : i \in S} \lambda(S) = 1$  for all  $i \in N$  such that  $\mathcal{B} = \{S \in 2^N \setminus \{\emptyset\} \mid \lambda(S) > 0\}$ .

A *transferable utility game* (cf. Von Neumann and Morgenstern (1944)) is a pair  $(N, v)$  in which  $v : 2^N \rightarrow \mathbb{R}$  is a characteristic function assigning to each coalition  $S \in 2^N$  its *worth*  $v(S) \in \mathbb{R}$  such that  $v(\emptyset) = 0$ . The number  $\frac{v(S)}{|S|}$  is called the *average worth* of  $S \in 2^N \setminus \{\emptyset\}$ . Let  $\text{TU}^N$  denote the class of all transferable utility games with player set  $N$ . For convenience, we abbreviate  $(N, v) \in \text{TU}^N$  to  $v \in \text{TU}^N$ . For any  $v \in \text{TU}^N$  the *subgame*  $v_S \in \text{TU}^S$  on  $S \in 2^N \setminus \{\emptyset\}$  is given by  $v_S(R) = v(R)$  for all  $R \in 2^S$ . A TU-game  $v \in \text{TU}^N$  is called

- *superadditive* if  $v(S) + v(T) \leq v(S \cup T)$  for all  $S, T \in 2^N$  for which  $S \cap T = \emptyset$ ;
- *convex* (cf. Shapley (1971)) if  $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$  for all  $S, T \in 2^N$ ;
- *balanced* (cf. Bondareva (1963) and Shapley (1967)) if  $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) v(S) \leq v(N)$  for every  $\lambda : 2^N \setminus \{\emptyset\} \rightarrow [0, 1]$  with  $\sum_{S \in 2^N : i \in S} \lambda(S) = 1$  for all  $i \in N$ .

Note that convexity implies superadditivity. Shapley (1971) showed that convexity implies balancedness. The *core* (cf. Gillies (1959)) of any  $v \in \text{TU}^N$  is given by

$$\mathcal{C}(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \forall S \in 2^N : \sum_{i \in S} x_i \geq v(S) \right\}.$$

Bondareva (1963) and Shapley (1967) showed that  $\mathcal{C}(v) \neq \emptyset$  if and only if  $v$  is balanced. A *solution* for transferable utility games  $f : \text{TU}^N \rightarrow \mathbb{R}^N$  assigns to any  $v \in \text{TU}^N$  a payoff allocation  $f(v) \in \mathbb{R}^N$ .

A *bankruptcy problem* (cf. O'Neill (1982)) is a triple  $(N, E, c)$  in which  $E \in \mathbb{R}$  is the *estate* and  $c \in \mathbb{R}^N$  is the vector of *claims* of  $N$  on  $E$  for which  $\sum_{i \in N} c_i \geq E$ . Note that the standard non-negativity conditions on  $E$  and  $c$  are dropped here for technical convenience later on. Let  $\text{BR}^N$  denote the class of all such bankruptcy problems with player set  $N$ . A *bankruptcy rule*  $f : \text{BR}^N \rightarrow \mathbb{R}^N$  assigns to any bankruptcy problem  $(N, E, c) \in \text{BR}^N$  a payoff allocation  $f(N, E, c) \in \mathbb{R}^N$  such that  $\sum_{i \in N} f_i(N, E, c) = E$  and  $f_i(N, E, c) \leq c_i$ . The *constrained equal awards rule*  $\text{CEA} : \text{BR}^N \rightarrow \mathbb{R}^N$  is for all  $(N, E, c) \in \text{BR}^N$  and any  $i \in N$  given by

$$\text{CEA}_i(N, E, c) = \min\{c_i, \alpha^{N, E, c}\},$$

in which  $\alpha^{N, E, c} = \min\{t \in \mathbb{R} \mid \sum_{i \in N} \min\{c_i, t\} = E\}$ .

### 3 The Egalitarian Procedure

In this section we introduce the egalitarian procedure for transferable utility games. This iterative procedure models negotiations between members of coalitions about the allocation of their worth, taking into account their coalitional egalitarian externalities. We formally define the egalitarian procedure after an illustrative example.

**Example 1.**

Let  $v \in \text{TU}^N$  be a transferable utility game with  $N = \{1, 2, 3\}$ . The table shows the worth of each coalition and the egalitarian distribution in each iteration of the egalitarian procedure.

$S$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	5	0	0	8	4	2	9
$\chi^{v,1}(S)$	$(5, \cdot, \cdot)$	$(\cdot, 0, \cdot)$	$(\cdot, \cdot, 0)$	$(4, 4, \cdot)$	$(2, \cdot, 2)$	$(\cdot, 1, 1)$	$(3, 3, 3)$
$\chi^{v,2}(S)$	$(5, \cdot, \cdot)$	$(\cdot, 0, \cdot)$	$(\cdot, \cdot, 0)$	$(5, 3, \cdot)$	$(5, \cdot, -1)$	$(\cdot, 1, 1)$	$(5, 2, 2)$
$\chi^{v,3}(S)$	$(5, \cdot, \cdot)$	$(\cdot, 3, \cdot)$	$(\cdot, \cdot, 0)$	$(5, 3, \cdot)$	$(5, \cdot, -1)$	$(\cdot, 3, -1)$	$(5, 3, 1)$
$\chi^{v,k}(S) (k \geq 4)$	$(5, \cdot, \cdot)$	$(\cdot, 3, \cdot)$	$(\cdot, \cdot, 1)$	$(5, 3, \cdot)$	$(5, \cdot, 1)$	$(\cdot, 3, 1)$	$(5, 3, 1)$

A natural way to start negotiating about the allocation of the worth of a coalition among its members is to divide it equally, i.e., in the first iteration, the egalitarian distribution  $\chi^{v,1}$  allocates in any coalition  $S \in 2^N \setminus \{\emptyset\}$  the average worth  $\frac{v(S)}{|S|}$  to each member  $i \in S$ . Players can only claim their highest allocated payoff if no other member of the corresponding coalition is allocated a higher payoff in any other coalition. All such players constitute the set of egalitarian claimants  $P^{v,1}$  with corresponding claims  $\gamma^{v,1}$ , and the coalitions in which they obtained their claims are contained in the collection of egalitarian admissible coalitions  $\mathcal{A}^{v,1}$ .

The highest payoff allocated by  $\chi^{v,1}$  to player 1 is 5 in coalition  $\{1\}$ , which player 1 can claim since this coalition contains no other members. The highest payoff allocated to player 2 is 4 in coalition  $\{1, 2\}$ , which player 2 cannot claim since player 1 is allocated a higher payoff in another coalition. The highest payoff allocated to player 3 is 3 in coalition  $\{1, 2, 3\}$ , which player 3 cannot claim since player 1 and 2 are allocated a higher payoff in other coalitions. This means that the set of 1-egalitarian claimants is given by  $P^{v,1} = \{1\}$ , the corresponding vector of 1-egalitarian claims is given by  $\gamma^{v,1} = (5, \cdot, \cdot)$ , and the collection of 1-egalitarian admissible coalitions is given by  $\mathcal{A}^{v,1} = \{\{1\}\}$ .

In a next iteration, the claimants claim their egalitarian claim in any coalition of which they are member and  $\chi^{v,2}$  divides the remaining worth equally among the other members. The claimants in  $P^{v,2}$  and their corresponding claims  $\gamma^{v,2}$  are constituted similarly to the first iteration, and  $\mathcal{A}^{v,2}$  contains the coalitions in which all members can obtain their claims. In this way, the players continue negotiating in further iterations. Note that, once a player has acquired an egalitarian claim, it remains fixed in all further iterations.

In particular, the highest payoff allocated by  $\chi^{v,2}$  to player 2 is 3 in coalition  $\{1, 2\}$ , which player 2 can claim since no other member is allocated a higher payoff in any other coalition. The highest payoff allocated to player 3 is 2 in coalition  $\{1, 2, 3\}$ , which player 3 cannot claim since player 2 is allocated a higher payoff in another coalition. This means that we have  $P^{v,2} = \{1, 2\}$ ,  $\gamma^{v,2} = (5, 3, \cdot)$  and  $\mathcal{A}^{v,2} = \{\{1\}, \{1, 2\}\}$ . In the third iteration, the highest payoff allocated by  $\chi^{v,3}$  to player 3 is 1 in coalition  $\{1, 2, 3\}$ , which player 3 can claim. We have  $P^{v,3} = \{1, 2, 3\}$ ,  $\gamma^{v,3} = (5, 3, 1)$  and  $\mathcal{A}^{v,3} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$ . Note that coalition  $\{2\}$  is not egalitarian admissible, since the egalitarian distribution allocates more than the worth of coalition  $\{2\}$ . In all further iterations, all players are allocated their claims in each coalition of which they are member, and the collection of egalitarian admissible coalitions remains unchanged.  $\triangle$

**Definition 1** (The Egalitarian Procedure).

Let  $v \in \text{TU}^N$  be a transferable utility game. The set of 0-egalitarian claimants is given by  $P^{v,0} = \emptyset$ . Let  $k \in \mathbb{N}$ . The  $k$ -egalitarian distribution is the function  $\chi^{v,k}$  assigning to each  $S \in 2^N \setminus \{\emptyset\}$  the payoff allocation  $\chi^{v,k}(S) \in \mathbb{R}^S$  given by

$$\chi_i^{v,k}(S) = \begin{cases} \gamma_i^{v,k-1} & \text{if } i \in S \cap P^{v,k-1}; \\ \frac{v(S) - \sum_{j \in S \cap P^{v,k-1}} \gamma_j^{v,k-1}}{|S \setminus P^{v,k-1}|} & \text{if } i \in S \setminus P^{v,k-1}. \end{cases}$$

The collection of  $k$ -egalitarian admissible coalitions is given by  $\mathcal{A}^{v,k} = \{S \in 2^N \setminus \{\emptyset\} \mid \sum_{i \in S} \chi_i^{v,k}(S) = v(S), \forall i \in S \forall T \in 2^N : i \in T : \chi_i^{v,k}(T) \leq \chi_i^{v,k}(S)\}$ . The set of  $k$ -egalitarian claimants  $P^{v,k} \in 2^N \setminus \{\emptyset\}$  is given by  $P^{v,k} = \bigcup_{S \in \mathcal{A}^{v,k}} S$ . The vector of  $k$ -egalitarian claims  $\gamma^{v,k} \in \mathbb{R}^{P^{v,k}}$  is for all  $i \in P^{v,k}$  given by  $\gamma_i^{v,k} = \chi_i^{v,k}(S)$ , where  $i \in S \in \mathcal{A}^{v,k}$ .

The payoff  $\chi_i^{v,k}(S)$  allocated to a player  $i \in S \setminus P^{v,k-1}$  is called the *average remaining worth* of  $S \in 2^N \setminus \{\emptyset\}$ . A typical observation is that a  $k$ -egalitarian distribution is in general overefficient, i.e. it possibly allocates more than the worth of a coalition.

**Lemma 3.1.**

Let  $v \in \text{TU}^N$  and let  $S \in 2^N \setminus \{\emptyset\}$ . Then  $\sum_{i \in S} \chi_i^{v,k}(S) \geq v(S)$  for all  $k \in \mathbb{N}$ . Moreover, if  $S \not\subseteq P^{v,k-1}$  for some  $k \in \mathbb{N}$ , then  $\sum_{i \in S} \chi_i^{v,k}(S) = v(S)$ .

*Proof.* We show the statement by induction on  $k$ . Using  $P^{v,0} = \emptyset$ , we can write

$$\sum_{i \in S} \chi_i^{v,1}(S) = \sum_{i \in S \setminus P^{v,0}} \left( \frac{v(S) - \sum_{j \in S \cap P^{v,0}} \gamma_j^{v,0}}{|S \setminus P^{v,0}|} \right) = \sum_{i \in S} \left( \frac{v(S)}{|S|} \right) = |S| \left( \frac{v(S)}{|S|} \right) = v(S).$$

Let  $k \in \mathbb{N}$  and assume that  $\sum_{i \in S} \chi_i^{v,k}(S) \geq v(S)$ . If  $S \subseteq P^{v,k}$ , then

$$\sum_{i \in S} \chi_i^{v,k+1}(S) = \sum_{i \in S} \gamma_i^{v,k} \geq \sum_{i \in S} \chi_i^{v,k}(S) \geq v(S).$$

If  $S \not\subseteq P^{v,k}$ , then

$$\begin{aligned} \sum_{i \in S} \chi_i^{v,k+1}(S) &= \sum_{i \in S \cap P^{v,k}} \chi_i^{v,k+1}(S) + \sum_{i \in S \setminus P^{v,k}} \chi_i^{v,k+1}(S) \\ &= \sum_{i \in S \cap P^{v,k}} \gamma_i^{v,k} + \sum_{i \in S \setminus P^{v,k}} \left( \frac{v(S) - \sum_{j \in S \cap P^{v,k}} \gamma_j^{v,k}}{|S \setminus P^{v,k}|} \right) \\ &= \sum_{i \in S \cap P^{v,k}} \gamma_i^{v,k} + |S \setminus P^{v,k}| \left( \frac{v(S) - \sum_{j \in S \cap P^{v,k}} \gamma_j^{v,k}}{|S \setminus P^{v,k}|} \right) \\ &= \sum_{i \in S \cap P^{v,k}} \gamma_i^{v,k} + v(S) - \sum_{j \in S \cap P^{v,k}} \gamma_j^{v,k} \\ &= v(S). \end{aligned}$$

□

By definition, only coalitions for which the egalitarian distribution allocates exactly the worth among its members can be egalitarian admissible. The question arises whether egalitarian admissible coalitions exist in each iteration for any transferable utility game. We show that in each iteration of the egalitarian procedure at least one extra player becomes an egalitarian claimant as long as the collection of egalitarian admissible coalitions is not a cover, which implies that egalitarian admissible coalitions indeed always exist.

**Lemma 3.2.**

Let  $v \in \text{TU}^N$ . Then  $\mathcal{A}^{v,k} \subseteq \mathcal{A}^{v,k+1}$  for all  $k \in \mathbb{N}$ . Moreover, if  $P^{v,k-1} \neq N$  for some  $k \in \mathbb{N}$ , then  $P^{v,k-1} \subset P^{v,k}$ .

*Proof.* Let  $k \in \mathbb{N}$  and assume that  $P^{v,k-1} \neq N$ . Let  $S \in 2^N$  with  $S \not\subseteq P^{v,k-1}$  be the coalition with highest average remaining worth. Then we know from Lemma 3.1 that  $\sum_{i \in S} \chi_i^{v,k}(S) = v(S)$ . Moreover, for all  $i \in S$  we have  $\chi_i^{v,k}(T) \leq \chi_i^{v,k}(S)$  for all  $T \in 2^N$  for which  $i \in T$ . This means that  $S \in \mathcal{A}^{v,k}$  and  $S \subseteq P^{v,k}$ .

Let  $k \in \mathbb{N}$  and let  $S \in \mathcal{A}^{v,k}$ . Then  $S \subseteq P^{v,k}$  and we can write

$$\sum_{i \in S} \chi_i^{v,k+1}(S) = \sum_{i \in S} \gamma_i^{v,k} = \sum_{i \in S} \chi_i^{v,k}(S) = v(S).$$

Moreover, for all  $i \in S$  we have  $\chi_i^{v,k+1}(T) \leq \chi_i^{v,k+1}(S)$  for all  $T \in 2^N$  for which  $i \in T$ . This means that  $S \in \mathcal{A}^{v,k+1}$ .  $\square$

Lemma 3.2 not only tells us that egalitarian admissible coalitions always exist, but also that the collection of egalitarian admissible coalitions weakly extends in each iteration. The structure of this collection is determined by the structure of the underlying transferable utility game. It turns out that well-known properties of TU-games have interesting implications for the structure of the collection of egalitarian admissible coalitions in each iteration. We derive those implications for superadditive, convex and balanced transferable utility games.

**Proposition 3.3.**

Let  $v \in \text{TU}^N$  and let  $k \in \mathbb{N}$ .

- (i) If  $v$  is superadditive, then  $S_1 \cup S_2 \in \mathcal{A}^{v,k}$  for all  $S_1, S_2 \in \mathcal{A}^{v,k}$  with  $S_1 \cap S_2 = \emptyset$ .
- (ii) If  $v$  is convex, then  $S_1 \cup S_2 \in \mathcal{A}^{v,k}$  and  $S_1 \cap S_2 \in \mathcal{A}^{v,k}$  for all  $S_1, S_2 \in \mathcal{A}^{v,k}$ .
- (iii) If  $v$  is balanced, then  $N \in \mathcal{A}^{v,k}$  if there exists a balanced collection  $\mathcal{B}$  with  $\mathcal{B} \subseteq \mathcal{A}^{v,k}$ .

*Proof.* (i) Assume that  $v$  is superadditive. Let  $S_1, S_2 \in \mathcal{A}^{v,k}$  with  $S_1 \cap S_2 = \emptyset$ . Then we know from Lemma 3.1 that  $\sum_{i \in S_1 \cup S_2} \chi_i^{v,k}(S_1 \cup S_2) \geq v(S_1 \cup S_2)$ . We can write

$$\begin{aligned} v(S_1 \cup S_2) &\geq v(S_1) + v(S_2) &&= \sum_{i \in S_1} \chi_i^{v,k}(S_1) + \sum_{i \in S_2} \chi_i^{v,k}(S_2) \\ &\geq \sum_{i \in S_1} \chi_i^{v,k}(S_1 \cup S_2) + \sum_{i \in S_2} \chi_i^{v,k}(S_1 \cup S_2) = \sum_{i \in S_1 \cup S_2} \chi_i^{v,k}(S_1 \cup S_2) \\ &\geq v(S_1 \cup S_2). \end{aligned}$$

This implies  $\sum_{i \in S_1 \cup S_2} \chi_i^{v,k}(S_1 \cup S_2) = v(S_1 \cup S_2)$ . Moreover, for all  $i \in S_1 \cup S_2$  we have  $\chi_i^{v,k}(T) \leq \chi_i^{v,k}(S_1 \cup S_2)$  for all  $T \in 2^N$  for which  $i \in T$ . Hence,  $S_1 \cup S_2 \in \mathcal{A}^{v,k}$ .



(ii) Assume that  $v$  is convex. Let  $S_1, S_2 \in \mathcal{A}^{v,k}$ . Then we know from Lemma 3.1 that  $\sum_{i \in S_1 \cup S_2} \chi_i^{v,k}(S_1 \cup S_2) \geq v(S_1 \cup S_2)$  and  $\sum_{i \in S_1 \cap S_2} \chi_i^{v,k}(S_1 \cap S_2) \geq v(S_1 \cap S_2)$ . We can write

$$\begin{aligned}
v(S_1 \cup S_2) + v(S_1 \cap S_2) &\geq v(S_1) + v(S_2) \\
&= \sum_{i \in S_1} \chi_i^{v,k}(S_1) + \sum_{i \in S_2} \chi_i^{v,k}(S_2) \\
&= \sum_{i \in S_1} \gamma_i^{v,k} + \sum_{i \in S_2} \gamma_i^{v,k} \\
&= \sum_{i \in S_1 \cup S_2} \gamma_i^{v,k} + \sum_{i \in S_1 \cap S_2} \gamma_i^{v,k} \\
&\geq \sum_{i \in S_1 \cup S_2} \chi_i^{v,k}(S_1 \cup S_2) + \sum_{i \in S_1 \cap S_2} \chi_i^{v,k}(S_1 \cap S_2) \\
&\geq v(S_1 \cup S_2) + v(S_1 \cap S_2).
\end{aligned}$$

This implies  $\sum_{i \in S_1 \cup S_2} \chi_i^{v,k}(S_1 \cup S_2) = v(S_1 \cup S_2)$  and  $\sum_{i \in S_1 \cap S_2} \chi_i^{v,k}(S_1 \cap S_2) = v(S_1 \cap S_2)$ . Moreover, for all  $i \in S_1 \cup S_2$  we have  $\chi_i^{v,k}(T) \leq \chi_i^{v,k}(S_1 \cup S_2)$  for all  $T \in 2^N$  for which  $i \in T$ , and for all  $i \in S_1 \cap S_2$  we have  $\chi_i^{v,k}(T) \leq \chi_i^{v,k}(S_1 \cap S_2)$  for all  $T \in 2^N$  for which  $i \in T$ . Hence,  $S_1 \cup S_2 \in \mathcal{A}^{v,k}$  and  $S_1 \cap S_2 \in \mathcal{A}^{v,k}$ .

(iii) Assume that  $v$  is balanced. Let  $\mathcal{B} \subseteq \mathcal{A}^{v,k}$  be a balanced collection and let  $\lambda : 2^N \setminus \{\emptyset\} \rightarrow [0, 1]$  with  $\sum_{S \in 2^N : i \in S} \lambda(S) = 1$  for all  $i \in N$  such that  $\mathcal{B} = \{S \in 2^N \setminus \{\emptyset\} \mid \lambda(S) > 0\}$ . Then we know from Lemma 3.1 that  $\sum_{i \in N} \chi_i^{v,k}(N) \geq v(N)$ . We can write

$$\begin{aligned}
v(N) &\geq \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) v(S) = \sum_{S \in \mathcal{B}} \lambda(S) v(S) = \sum_{S \in \mathcal{B}} \lambda(S) \sum_{i \in S} \chi_i^{v,k}(S) \\
&= \sum_{i \in N} \sum_{S \in \mathcal{B} : i \in S} \lambda(S) \chi_i^{v,k}(S) \geq \sum_{i \in N} \sum_{S \in \mathcal{B} : i \in S} \lambda(S) \chi_i^{v,k}(N) = \sum_{i \in N} \chi_i^{v,k}(N) \sum_{S \in \mathcal{B} : i \in S} \lambda(S) \\
&= \sum_{i \in N} \chi_i^{v,k}(N) \sum_{S \in 2^N : i \in S} \lambda(S) = \sum_{i \in N} \chi_i^{v,k}(N) \geq v(N).
\end{aligned}$$

This implies  $\sum_{i \in N} \chi_i^{v,k}(N) = v(N)$ . Moreover, for all  $i \in N$  we have  $\chi_i^{v,k}(T) \leq \chi_i^{v,k}(N)$  for all  $T \in 2^N$  for which  $i \in T$ . Hence,  $N \in \mathcal{A}^{v,k}$ .  $\square$

The egalitarian procedure is an egalitarian bargaining model that takes participation constraints explicitly into account. The egalitarian admissible coalitions can be considered as the coalitions in which members prefer to participate, concerning the corresponding allocation prescribed by the egalitarian distribution. This consideration suggests that the assigned allocation, consisting of the egalitarian claim for each member, is stable against subcoalitional deviations. Indeed, the vector of egalitarian claims corresponding to the members of an egalitarian admissible coalition is an element of the core of the induced subgame.

**Proposition 3.4.**

Let  $v \in \text{TU}^N$  and let  $k \in \mathbb{N}$ . Then  $(\gamma_i^{v,k})_{i \in S} \in \mathcal{C}(v_S)$  for all  $S \in \mathcal{A}^{v,k}$ .

*Proof.* Let  $S \in \mathcal{A}^{v,k}$ . Then we can write

$$\sum_{i \in S} \gamma_i^{v,k} = \sum_{i \in S} \chi_i^{v,k}(S) = v(S) = v_S(S).$$

Moreover, using Lemma 3.1, we can write for all  $R \in 2^S$

$$\sum_{i \in R} \gamma_i^{v,k} = \sum_{i \in R} \chi_i^{v,k}(S) \geq \sum_{i \in R} \chi_i^{v,k}(R) \geq v(R) = v_S(R).$$

Hence,  $(\gamma_i^{v,k})_{i \in S} \in \mathcal{C}(v_S)$ .  $\square$

The egalitarian procedure reaches a steady state when the collection of egalitarian admissible coalitions is a cover, i.e. all players have become egalitarian claimants. From Lemma 3.2 we know that the egalitarian procedure converges to this steady state within a number of iterations which is bounded by the number of players in the underlying transferable utility game.

The players stop negotiating when they all have acquired an egalitarian claim. Although this egalitarian claim is bounded from below by the individual worth of the player, it is possibly negative. Nevertheless, the egalitarian claims can be obtained in one or more egalitarian admissible coalitions. They form aspiration levels for the allocation of the worth of the grand coalition. In the next section we further describe the egalitarian steady state and we define the procedural egalitarian solution which allocates the worth of the grand coalition in an egalitarian way among the players, taking into account their (generally overefficient) egalitarian claims.

## 4 The Procedural Egalitarian Solution

In this section we introduce the procedural egalitarian solution for transferable utility games. This solution is based on the egalitarian steady state to which the egalitarian procedure converges.

### Definition 2.

Let  $v \in \text{TU}^N$  be a transferable utility game. The *egalitarian steady state iteration*  $n^v \in \{1, \dots, |N|\}$  is given by  $n^v = \min\{k \in \mathbb{N} \mid P^{v,k} = N\}$ . The vector of *egalitarian claims*  $\hat{\gamma}^v \in \mathbb{R}^N$  is for all  $i \in N$  given by  $\hat{\gamma}_i^v = \gamma_i^{v,n^v}$ . The collection of *maximal egalitarian admissible coalitions*  $\hat{\mathcal{A}}^v \subseteq 2^N \setminus \{\emptyset\}$  is given by  $\hat{\mathcal{A}}^v = \{S \in \mathcal{A}^{v,n^v} \mid \forall T \in \mathcal{A}^{v,n^v} : S \not\subseteq T\}$ . The set of *strong egalitarian claimants*  $D^v \in 2^N$  is given by  $D^v = \bigcap_{S \in \hat{\mathcal{A}}^v} S$ .

Note that  $\sum_{i \in S} \hat{\gamma}_i^v \geq v(S)$  for all  $S \in 2^N$  and  $\mathcal{A}^{v,n^v} = \{S \in 2^N \setminus \{\emptyset\} \mid \sum_{i \in S} \hat{\gamma}_i^v = v(S)\}$ . Players can obtain their egalitarian claim in the egalitarian admissible coalitions of which they are member. We only consider the inclusion-wise maximal egalitarian admissible coalitions. Players which are member of all maximal egalitarian admissible coalitions are called *strong* egalitarian claimants. The procedural egalitarian solution assigns to the strong egalitarian claimants their claims, and divides the remaining worth of the grand coalition among the other players according to the constrained equal awards rule, the standard concept of egalitarianism in the context of bankruptcy problems.

### Definition 3 (The Procedural Egalitarian Solution).

The *procedural egalitarian solution*  $\Gamma : \text{TU}^N \rightarrow \mathbb{R}^N$  is for all  $v \in \text{TU}^N$  and any  $i \in N$  given by

$$\Gamma_i(v) = \begin{cases} \hat{\gamma}_i^v & \text{if } i \in D^v; \\ \text{CEA}_i \left( N \setminus D^v, v(N) - \sum_{j \in D^v} \hat{\gamma}_j^v, (\hat{\gamma}_j^v)_{j \in N \setminus D^v} \right) & \text{if } i \in N \setminus D^v. \end{cases}$$

Note that an interesting situation arises when the grand coalition is egalitarian admissible and consequently all players are strong egalitarian claimants. In such situation, the underlying transferable utility game is called *egalitarian stable*.

**Definition 4** (Egalitarian Stability).

A transferable utility game  $v \in \text{TU}^N$  is called *egalitarian stable* if  $\hat{\mathcal{A}}^v = \{N\}$ .

If  $v \in \text{TU}^N$  is egalitarian stable, we have  $D^v = N$  and  $\Gamma(v) = \hat{\gamma}^v$ . Moreover, from Proposition 3.4 we know that  $\Gamma(v) \in \mathcal{C}(v)$  if and only if  $v$  is egalitarian stable. The question arises whether we can formulate conditions on TU-games which imply egalitarian stability. Since the collection of egalitarian admissible coalitions is a cover, we know from Proposition 3.3 that convexity is such a condition. Clearly, balancedness is a necessary condition for egalitarian stability and from Proposition 3.3 we know that it is sufficient if there exists a balanced collection of egalitarian admissible coalitions.

The following example studies egalitarian stability for the class of glove games and derives the procedural egalitarian solution for a game for which the constrained egalitarian solution of Dutta and Ray (1989) does not exist.

**Example 2** (Glove Games).

In a glove game  $v \in \text{TU}^N$  there exist  $L, R \in 2^N \setminus \{\emptyset\}$  such that  $N = L \cup R$  and  $L \cap R = \emptyset$ . Players in  $L$  are endowed with a left-hand glove and players in  $R$  are endowed with a right-hand glove, but only pairs of one left-hand and one right-hand glove have value. The worth of a coalition  $S \in 2^N$  can therefore be described by  $v(S) = \min\{|L \cap S|, |R \cap S|\}$ . In a glove game, the egalitarian steady state is reached in the first iteration, i.e.  $n^v = 1$ . Moreover, we have  $\hat{\mathcal{A}}^v = \{S \in 2^N \mid v(S) = v(N), |L \cap S| = |R \cap S|\}$  and  $\hat{\gamma}_i^v = \frac{1}{2}$  for all  $i \in N$ . Consequently,

$$D^v = \begin{cases} L & \text{if } |L| < |R|; \\ N & \text{if } |L| = |R|; \\ R & \text{if } |L| > |R|. \end{cases}$$

This means that a glove game is egalitarian stable if and only if  $|L| = |R|$ . The procedural egalitarian solution divides a half per pair of gloves equally among the left-hand glove players, and the other half per pair of gloves equally among the right-hand glove players.

Let  $v \in \text{TU}^N$  be a glove game with  $L = \{1\}$  and  $R = \{2, 3\}$ . The table shows the worth of each coalition and the egalitarian distribution in the first iteration of the egalitarian procedure.

$S$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	0	0	0	1	1	0	1
$\chi^{v,1}(S)$	$(0, \cdot, \cdot)$	$(\cdot, 0, \cdot)$	$(\cdot, \cdot, 0)$	$(\frac{1}{2}, \frac{1}{2}, \cdot)$	$(\frac{1}{2}, \cdot, \frac{1}{2})$	$(\cdot, 0, 0)$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

We have  $\hat{\mathcal{A}}^{v,1} = \{\{1, 2\}, \{1, 3\}\}$ ,  $P^{v,1} = N$  and  $\gamma^{v,1} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ . This means that  $\hat{\mathcal{A}}^v = \{\{1, 2\}, \{1, 3\}\}$ ,  $D^v = \{1\}$  and  $\hat{\gamma}^v = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ . Consequently,  $\Gamma(v) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ .  $\triangle$

Next, we study some basic properties of the procedural egalitarian solution. We show that the procedural egalitarian solution satisfies efficiency, symmetry, dummy invariance and weak covariance. The properties are derived in the Appendix. Note that the procedural egalitarian solution is not relative invariant with respect to strategic equivalence. In fact, there does not exist a solution concept which satisfies relative invariance with respect to strategic equivalence and coincides with the constrained egalitarian solution on the class of convex games. We refer to Dutta and Ray (1989) for a discussion on why egalitarian solution concepts actually *should* fail to satisfy this stronger covariance property.

**Definition 5** (Elementary Properties).

A solution for transferable utility games  $f : \text{TU}^N \rightarrow \mathbb{R}^N$  satisfies

- *efficiency* if for all  $v \in \text{TU}^N$  we have  $\sum_{i \in N} f_i(v) = v(N)$ ;
- *symmetry* if for all  $v \in \text{TU}^N$  we have  $f_i(v) = f_j(v)$  for all  $i, j \in N$  for which  $v(\sigma^{i,j}(S)) = v(S)$  for all  $S \in 2^N$ ;
- *dummy invariance* if for all  $v \in \text{TU}^N$  we have  $f_i(v) = v(\{i\})$  for all  $i \in N$  for which  $v(S \cup \{i\}) = v(S) + v(\{i\})$  for all  $S \in 2^N$  with  $i \notin S$ ;
- *weak covariance* if for all  $v \in \text{TU}^N$  and any  $a \in \mathbb{R}_{++}$ ,  $b \in \mathbb{R}$  we have  $f_i(v') = af_i(v) + b$  for all  $i \in N$ , where  $v' \in \text{TU}^N$  is given by  $v'(S) = av(S) + b|S|$  for all  $S \in 2^N$ .

**Theorem 4.1.**

*The procedural egalitarian solution satisfies efficiency, symmetry, dummy invariance, and weak covariance.*

We conclude this section with an analysis of the procedural egalitarian solution on the class of convex games. We know that convex games are egalitarian stable. Moreover, the constrained egalitarian solution of Dutta and Ray (1989) can be computed algorithmically for convex games, as described by the following definition.

**Definition 6** (The Constrained Egalitarian Solution (cf. Dutta and Ray (1989))).

Let  $v \in \text{TU}^N$  be a convex transferable utility game. Let  $v_0 = v$  and  $T_0^v = \emptyset$ . For any  $k \in \mathbb{N}$ , let  $v_k$  assign to each  $S \subseteq N \setminus (\bigcup_{s=0}^{k-1} T_s^v)$  the worth  $v_k(S) = v_{k-1}(S \cup T_{k-1}^v) - v_{k-1}(T_{k-1}^v)$ , where  $T_k^v \in 2^N \setminus \{\emptyset\}$  is the largest coalition having the highest average worth in  $v_k$ . For any  $k \in \mathbb{N}$ , and any  $i \in T_k^v$ , the *constrained egalitarian solution* CES is given by  $\text{CES}_i(v) = \frac{v_k(T_k^v)}{|T_k^v|}$ .

**Theorem 4.2.**

*The procedural egalitarian solution coincides with the constrained egalitarian solution of Dutta and Ray (1989) on the class of convex games.*

*Proof.* Let  $v \in \text{TU}^N$  be a convex transferable utility game. Since  $v$  is egalitarian stable, we have  $\Gamma(v) = \hat{\gamma}^v$ .

We show by induction on  $k$  that we have  $v_k(S) = v(S \cup P^{v,k-1}) - \sum_{j \in P^{v,k-1}} \gamma_j^{v,k-1}$  for all  $S \subseteq N \setminus (\bigcup_{s=0}^{k-1} T_s^v)$ ,  $P^{v,k} = \bigcup_{s=0}^k T_s^v$ , and  $\text{CES}_i(v) = \hat{\gamma}_i^v$  for all  $i \in P^{v,k}$ .

From Proposition 3.3 we know that for all  $k \in \mathbb{N}$  we have  $S_1 \cup S_2 \in \mathcal{A}^{v,k}$  for all  $S_1, S_2 \in \mathcal{A}^{v,k}$ . In particular, this implies that  $P^{v,k} \in \mathcal{A}^{v,k}$  for all  $k \in \mathbb{N}$ .

Using  $v_0 = v$ ,  $T_0^v = \emptyset$  and  $P^{v,0} = \emptyset$ , we can write for all  $S \subseteq N$

$$v_1(S) = v_0(S \cup T_0^v) - v_0(T_0^v) = v(S) - v(\emptyset) = v(S) = v(S \cup P^{v,0}) - \sum_{j \in P^{v,0}} \gamma_j^{v,0}.$$

For all  $S_1, S_2 \in \mathcal{A}^{v,1}$ , and any  $i \in S_1$  and  $j \in S_2$ , we can write

$$\frac{v(S_1)}{|S_1|} = \chi_i^{v,1}(S_1) = \chi_i^{v,1}(S_1 \cup S_2) = \frac{v(S_1 \cup S_2)}{|S_1 \cup S_2|} = \chi_j^{v,1}(S_1 \cup S_2) = \chi_j^{v,1}(S_2) = \frac{v(S_2)}{|S_2|}.$$

This means that each 1-egalitarian admissible coalition has the highest average worth. Since  $P^{v,1} \in \mathcal{A}^{v,1}$ , then  $P^{v,1}$  is the largest coalition with highest average worth in  $v_1$ . Hence,  $v_1(S) = v(S \cup P^{v,0}) - \sum_{j \in P^{v,0}} \gamma_j^{v,0}$  for all  $S \subseteq N \setminus T_0^v$ ,  $P^{v,1} = T_1^v$ , and  $\text{CES}_i(v) = \hat{\gamma}_i^v$  for all  $i \in P^{v,1}$ .

Let  $k \in \mathbb{N}$  and assume that we have  $v_k(S) = v(S \cup P^{v,k-1}) - \sum_{j \in P^{v,k-1}} \gamma_j^{v,k-1}$  for all  $S \subseteq N \setminus (\bigcup_{s=0}^{k-1} T_s^v)$ ,  $P^{v,l} = \bigcup_{s=0}^l T_s^v$  for all  $l \in \{1, \dots, k\}$ , and  $\text{CES}_i(v) = \hat{\gamma}_i^v$  for all  $i \in P^{v,k}$ . We can write for all  $S \subseteq N \setminus (\bigcup_{s=0}^k T_s^v)$

$$\begin{aligned} v_{k+1}(S) &= v_k(S \cup T_k^v) - v_k(T_k^v) \\ &= v(S \cup T_k^v \cup P^{v,k-1}) - \sum_{j \in P^{v,k-1}} \gamma_j^{v,k-1} - v(T_k^v \cup P^{v,k-1}) + \sum_{j \in P^{v,k-1}} \gamma_j^{v,k-1} \\ &= v(S \cup P^{v,k}) - v(P^{v,k}) \\ &= v(S \cup P^{v,k}) - \sum_{j \in P^{v,k}} \gamma_j^{v,k}, \end{aligned}$$

where the last equality follows from  $P^{v,k} \in \mathcal{A}^{v,k}$ . This means that for all  $S \subseteq N \setminus (\bigcup_{s=0}^k T_s^v)$  we have for all  $i \in S$

$$\begin{aligned} \frac{v_{k+1}(S)}{|S|} &= \frac{v(S \cup P^{v,k}) - \sum_{j \in P^{v,k}} \gamma_j^{v,k}}{|S|} \\ &= \frac{v(S \cup P^{v,k}) - \sum_{j \in (S \cup P^{v,k}) \cap P^{v,k}} \gamma_j^{v,k}}{|(S \cup P^{v,k}) \setminus P^{v,k}|} = \chi_i^{v,k+1}(S \cup P^{v,k}). \end{aligned}$$

From Lemma 3.2 and Proposition 3.3 we know that  $S \cup P^{v,k} \in \mathcal{A}^{v,k+1}$  for all  $S \in \mathcal{A}^{v,k+1}$ . For all  $S_1, S_2 \in \mathcal{A}^{v,k+1}$ , and any  $i \in S_1 \setminus P^{v,k}$  and  $j \in S_2 \setminus P^{v,k}$ , we can write

$$\chi_i^{v,k+1}(S_1) = \chi_i^{v,k+1}(S_1 \cup S_2) = \chi_j^{v,k+1}(S_1 \cup S_2) = \chi_j^{v,k+1}(S_2).$$

This means that each coalition in  $\mathcal{A}^{v,k+1} \setminus \mathcal{A}^{v,k}$  has the highest average remaining worth. Then  $P^{v,k+1} \setminus P^{v,k}$  is the largest coalition with highest average worth in  $v_{k+1}$ .

This means that we have  $v_{k+1}(S) = v(S \cup P^{v,k}) - \sum_{j \in P^{v,k}} \gamma_j^{v,k}$  for all  $S \subseteq N \setminus (\bigcup_{s=0}^k T_s^v)$ ,  $P^{v,k+1} = \bigcup_{s=0}^{k+1} T_s^v$ , and  $\text{CES}_i(v) = \hat{\gamma}_i^v$  for all  $i \in P^{v,k+1}$ .  $\square$

**Example 3** (Bankruptcy Games).

In a nonnegative bankruptcy problem  $(N, E, c) \in \text{BR}^N$  we have  $E \geq 0$  and  $c \in \mathbb{R}_+^N$ . The bankruptcy game  $v^{E,c} \in \text{TU}^N$  (cf. O'Neill (1982)) corresponding to the nonnegative bankruptcy problem  $(N, E, c)$  is given by  $v^{E,c}(S) = \max\{E - \sum_{i \in N \setminus S} c_i, 0\}$  for all  $S \in 2^N$ . Curiel, Maschler, and Tijs (1987) showed that bankruptcy games are convex. This means that the procedural egalitarian solution of a bankruptcy game coincides with the constrained egalitarian solution of Dutta and Ray (1989), which equals the constrained equal awards rule of the underlying bankruptcy problem.

Let  $(N, E, c) \in \text{BR}^N$  be a bankruptcy problem with  $N = \{1, 2, 3\}$ ,  $E = 12$  and  $c = (2, 6, 8)$ . Then we have  $\text{CEA}(N, E, c) = (2, 5, 5)$ . The table shows the worth of each coalition in the corresponding bankruptcy game  $v^{E,c} \in \text{TU}^N$  and the egalitarian distribution in the first two iterations of the egalitarian procedure.

$S$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v^{E,c}(S)$	0	2	4	4	6	10	12
$\chi^{v^{E,c},1}(S)$	$(0, \cdot, \cdot)$	$(\cdot, 2, \cdot)$	$(\cdot, \cdot, 4)$	$(2, 2, \cdot)$	$(3, \cdot, 3)$	$(\cdot, 5, 5)$	$(4, 4, 4)$
$\chi^{v^{E,c},2}(S)$	$(0, \cdot, \cdot)$	$(\cdot, 5, \cdot)$	$(\cdot, \cdot, 5)$	$(-1, 5, \cdot)$	$(1, \cdot, 5)$	$(\cdot, 5, 5)$	$(2, 5, 5)$

In the first iteration we have  $\mathcal{A}^{v,1} = \{\{2, 3\}\}$ ,  $P^{v,1} = \{2, 3\}$  and  $\gamma^{v,1} = (\cdot, 5, 5)$ . In the second iteration we have  $\mathcal{A}^{v,2} = \{\{2, 3\}, \{1, 2, 3\}\}$ ,  $P^{v,2} = N$  and  $\gamma^{v,2} = (2, 5, 5)$ . This means that  $\hat{\mathcal{A}}^v = \{N\}$ ,  $D^v = N$  and  $\hat{\gamma}^v = (2, 5, 5)$ . Consequently,  $\Gamma(v) = (2, 5, 5)$ .  $\triangle$

The next result follows immediately from the elucidation in Example 3. We refer to Dietzenbacher (2014) for a direct proof.

**Corollary 4.3.**

*The procedural egalitarian solution of a bankruptcy game coincides with the constrained equal awards rule of the underlying bankruptcy problem.*

## 5 Concluding Remarks

In this section we formulate some concluding remarks on the procedural egalitarian solution and some suggestions for future research. We know that the procedural egalitarian solution is single-valued and exists for any TU-game. Moreover, from Theorem 4.2 we know that it coincides with the well-known constrained egalitarian solution of Dutta and Ray (1989) on the class of convex games. This means that the procedural egalitarian solution for convex games is axiomatically characterized by Dutta (1990), Klijn et al. (2000) and Arin et al. (2003). Future research could look for properties which extend axiomatic characterizations of the procedural egalitarian solution for convex games to more general classes of games.

Transferable utility games for which the egalitarian solution is an element of the core are called egalitarian stable. We know that convexity is a sufficient condition for a TU-game to be egalitarian stable. Example 1 shows that this condition is not necessary. Balancedness is trivially a necessary condition for a TU-game to be egalitarian stable. Example 2 shows that this condition is not sufficient. The class of egalitarian stable games is the class of TU-games for which the grand coalition is egalitarian admissible. Whether the grand coalition is egalitarian admissible can be determined by applying the egalitarian procedure. Future research could look for a characterization of the class of egalitarian stable transferable utility games. This will contribute to a better understanding of situations in which egalitarianism and coalitional rationality do not conflict.

## Appendix

**Lemma A.1.**

*The procedural egalitarian solution satisfies symmetry.*

*Proof.* Let  $v \in \text{TU}^N$  and let  $i_1, i_2 \in N$  such that  $v(\sigma^{i_1, i_2}(S)) = v(S)$  for all  $S \in 2^N$ . For all  $S \in 2^N$  for which  $i_1 \in S$  we have

$$\chi_{i_1}^{v,1}(S) = \frac{v(S)}{|S|} = \frac{v(\sigma^{i_1, i_2}(S))}{|\sigma^{i_1, i_2}(S)|} = \chi_{i_2}^{v,1}(\sigma^{i_1, i_2}(S)).$$

Using Lemma 3.1, we can write for all  $S \in \mathcal{A}^{v,1}$  for which  $i_1 \in S$  and  $i_2 \notin S$

$$\begin{aligned} v(\sigma^{i_1, i_2}(S)) &= v(S) = \sum_{i \in S \setminus \{i_1\}} \chi_i^{v,1}(S) + \chi_{i_1}^{v,1}(S) \\ &\geq \sum_{i \in \sigma^{i_1, i_2}(S) \setminus \{i_2\}} \chi_i^{v,1}(\sigma^{i_1, i_2}(S)) + \chi_{i_2}^{v,1}(\sigma^{i_1, i_2}(S)) \\ &\geq v(\sigma^{i_1, i_2}(S)). \end{aligned}$$

This implies  $\mathcal{A}^{v,1} = \{\sigma^{i_1,i_2}(S) \mid S \in \mathcal{A}^{v,1}\}$ . Consequently,

$$P^{v,1} = \bigcup_{S \in \mathcal{A}^{v,1}} S = \bigcup_{S \in \mathcal{A}^{v,1}} \sigma^{i_1,i_2}(S) = \sigma^{i_1,i_2} \left( \bigcup_{S \in \mathcal{A}^{v,1}} S \right) = \sigma^{i_1,i_2}(P^{v,1}).$$

If  $i_1 \in P^{v,1}$ , then we have for all  $S \in \mathcal{A}^{v,1}$  with  $i_1 \in S$

$$\gamma_{i_1}^{v,1} = \chi_{i_1}^{v,1}(S) = \chi_{i_2}^{v,1}(\sigma^{i_1,i_2}(S)) = \gamma_{i_2}^{v,1}.$$

Let  $k \in \mathbb{N}$  and assume that  $\chi_{i_1}^{v,k}(S) = \chi_{i_2}^{v,k}(\sigma^{i_1,i_2}(S))$  for all  $S \in 2^N \setminus \{\emptyset\}$  for which  $i_1 \in S$ ,  $\mathcal{A}^{v,k} = \{\sigma^{i_1,i_2}(S) \mid S \in \mathcal{A}^{v,k}\}$ ,  $P^{v,k} = \sigma^{i_1,i_2}(P^{v,k})$  and  $\gamma_{i_1}^{v,k} = \gamma_{i_2}^{v,k}$  if  $i_1 \in P^{v,k}$ . Then we can write for all  $S \in 2^N$  for which  $i_1 \in S$

$$\begin{aligned} \chi_{i_1}^{v,k+1}(S) &= \begin{cases} \gamma_{i_1}^{v,k} & \text{if } i_1 \in S \cap P^{v,k}; \\ \frac{v(S) - \sum_{j \in S \cap P^{v,k}} \gamma_j^{v,k}}{|S \setminus P^{v,k}|} & \text{if } i_1 \in S \setminus P^{v,k} \end{cases} \\ &= \begin{cases} \gamma_{i_2}^{v,k} & \text{if } i_2 \in \sigma^{i_1,i_2}(S) \cap P^{v,k}; \\ \frac{v(\sigma^{i_1,i_2}(S)) - \sum_{j \in \sigma^{i_1,i_2}(S) \cap P^{v,k}} \gamma_j^{v,k}}{|\sigma^{i_1,i_2}(S) \setminus P^{v,k}|} & \text{if } i_2 \in S \setminus P^{v,k} \end{cases} \\ &= \chi_{i_2}^{v,k+1}(\sigma^{i_1,i_2}(S)). \end{aligned}$$

Using Lemma 3.1, we can write for all  $S \in \mathcal{A}^{v,k+1}$  for which  $i_1 \in S$  and  $i_2 \notin S$

$$\begin{aligned} v(\sigma^{i_1,i_2}(S)) &= v(S) = \sum_{i \in S \setminus \{i_1\}} \chi_i^{v,k+1}(S) + \chi_{i_1}^{v,k+1}(S) \\ &\geq \sum_{i \in \sigma^{i_1,i_2}(S) \setminus \{i_2\}} \chi_i^{v,k+1}(\sigma^{i_1,i_2}(S)) + \chi_{i_2}^{v,k+1}(\sigma^{i_1,i_2}(S)) \\ &\geq v(\sigma^{i_1,i_2}(S)). \end{aligned}$$

This implies  $\mathcal{A}^{v,k+1} = \{\sigma^{i_1,i_2}(S) \mid S \in \mathcal{A}^{v,k+1}\}$ . Consequently,

$$P^{v,k+1} = \bigcup_{S \in \mathcal{A}^{v,k+1}} S = \bigcup_{S \in \mathcal{A}^{v,k+1}} \sigma^{i_1,i_2}(S) = \sigma^{i_1,i_2} \left( \bigcup_{S \in \mathcal{A}^{v,k+1}} S \right) = \sigma^{i_1,i_2}(P^{v,k+1}).$$

If  $i_1 \in P^{v,k+1}$ , then we have for all  $S \in \mathcal{A}^{v,k+1}$  with  $i_1 \in S$

$$\gamma_{i_1}^{v,k+1} = \chi_{i_1}^{v,k+1}(S) = \chi_{i_2}^{v,k+1}(\sigma^{i_1,i_2}(S)) = \gamma_{i_2}^{v,k+1}.$$

This means that for all  $k \in \mathbb{N}$  we have  $\chi_{i_1}^{v,k}(S) = \chi_{i_2}^{v,k}(\sigma^{i_1,i_2}(S))$  for all  $S \in 2^N \setminus \{\emptyset\}$  for which  $i_1 \in S$ ,  $\mathcal{A}^{v,k} = \{\sigma^{i_1,i_2}(S) \mid S \in \mathcal{A}^{v,k}\}$ ,  $P^{v,k} = \sigma^{i_1,i_2}(P^{v,k})$  and  $\gamma_{i_1}^{v,k} = \gamma_{i_2}^{v,k}$  if  $i_1 \in P^{v,k}$ . Then we have  $\hat{\mathcal{A}}^v = \{\sigma^{i_1,i_2}(S) \mid S \in \hat{\mathcal{A}}^v\}$  and

$$D^v = \bigcap_{S \in \hat{\mathcal{A}}^v} S = \bigcap_{S \in \hat{\mathcal{A}}^v} \sigma^{i_1,i_2}(S) = \sigma^{i_1,i_2} \left( \bigcap_{S \in \hat{\mathcal{A}}^v} S \right) = \sigma^{i_1,i_2}(D^v).$$

Moreover, we have

$$\hat{\gamma}_{i_1}^v = \gamma_{i_1}^{v,n^v} = \gamma_{i_2}^{v,n^v} = \hat{\gamma}_{i_2}^v.$$

Then we can write

$$\begin{aligned}
\Gamma_{i_1}(v) &= \begin{cases} \hat{\gamma}_{i_1}^v & \text{if } i_1 \in D^v; \\ \text{CEA}_{i_1} \left( N \setminus D^v, v(N) - \sum_{j \in D^v} \hat{\gamma}_j^v, (\hat{\gamma}_j^v)_{j \in N \setminus D^v} \right) & \text{if } i_1 \in N \setminus D^v \end{cases} \\
&= \begin{cases} \hat{\gamma}_{i_1}^v & \text{if } i_1 \in D^v; \\ \min \left\{ \hat{\gamma}_{i_1}^v, \alpha^{N \setminus D^v, v(N) - \sum_{j \in D^v} \hat{\gamma}_j^v, (\hat{\gamma}_j^v)_{j \in N \setminus D^v}} \right\} & \text{if } i_1 \in N \setminus D^v \end{cases} \\
&= \begin{cases} \hat{\gamma}_{i_2}^v & \text{if } i_2 \in D^v; \\ \min \left\{ \hat{\gamma}_{i_2}^v, \alpha^{N \setminus D^v, v(N) - \sum_{j \in D^v} \hat{\gamma}_j^v, (\hat{\gamma}_j^v)_{j \in N \setminus D^v}} \right\} & \text{if } i_2 \in N \setminus D^v \end{cases} \\
&= \begin{cases} \hat{\gamma}_{i_2}^v & \text{if } i_2 \in D^v; \\ \text{CEA}_{i_2} \left( N \setminus D^v, v(N) - \sum_{j \in D^v} \hat{\gamma}_j^v, (\hat{\gamma}_j^v)_{j \in N \setminus D^v} \right) & \text{if } i_2 \in N \setminus D^v \end{cases} \\
&= \Gamma_{i_2}(v).
\end{aligned}$$

Hence, the procedural egalitarian solution satisfies symmetry.  $\square$

**Lemma A.2.**

*The procedural egalitarian solution satisfies dummy invariance.*

*Proof.* Let  $v \in \text{TU}^N$  and let  $i \in N$  such that  $v(S \cup \{i\}) = v(S) + v(\{i\})$  for all  $S \in 2^N$  for which  $i \notin S$ . For all  $S \in \mathcal{A}^{v, n^v}$  for which  $i \in S$  we can write

$$v(S \setminus \{i\}) = v(S) - v(\{i\}) = \sum_{j \in S} \hat{\gamma}_j^v - v(\{i\}) \geq \sum_{j \in S} \hat{\gamma}_j^v - \hat{\gamma}_i^v = \sum_{j \in S \setminus \{i\}} \hat{\gamma}_j^v \geq v(S \setminus \{i\}).$$

This implies  $\hat{\gamma}_i^v = v(\{i\})$ . For all  $S \in \mathcal{A}^{v, n^v}$  for which  $i \notin S$  we can write

$$v(S \cup \{i\}) = v(S) + v(\{i\}) = \sum_{j \in S} \hat{\gamma}_j^v + \hat{\gamma}_i^v = \sum_{j \in S \cup \{i\}} \hat{\gamma}_j^v.$$

This implies  $S \cup \{i\} \in \mathcal{A}^{v, n^v}$ . This means that  $i \in D^v$ , so we have  $\Gamma_i(v) = \hat{\gamma}_i^v = v(\{i\})$ . Hence, the procedural egalitarian solution satisfies dummy invariance.  $\square$

**Lemma A.3.**

*The procedural egalitarian solution satisfies weak covariance.*

*Proof.* Let  $v \in \text{TU}^N$ , let  $a \in \mathbb{R}_{++}$  and let  $b \in \mathbb{R}$ . Define  $v' \in \text{TU}^N$  by  $v'(S) = av(S) + b|S|$  for all  $S \in 2^N$ . For all  $S \in 2^N \setminus \{\emptyset\}$  and any  $i \in S$  we can write

$$\chi_i^{v', 1}(S) = \frac{v'(S)}{|S|} = \frac{av(S) + b|S|}{|S|} = a \frac{v(S)}{|S|} + b = a\chi_i^{v, 1}(S) + b.$$

Then we have

$$\begin{aligned}
\mathcal{A}^{v', 1} &= \{S \in 2^N \setminus \{\emptyset\} \mid \forall i \in S \forall T \in 2^N : i \in T : \chi_i^{v', 1}(T) \leq \chi_i^{v', 1}(S)\} \\
&= \{S \in 2^N \setminus \{\emptyset\} \mid \forall i \in S \forall T \in 2^N : i \in T : a\chi_i^{v, 1}(T) + b \leq a\chi_i^{v, 1}(S) + b\} \\
&= \{S \in 2^N \setminus \{\emptyset\} \mid \forall i \in S \forall T \in 2^N : i \in T : \chi_i^{v, 1}(T) \leq \chi_i^{v, 1}(S)\} \\
&= \mathcal{A}^{v, 1}.
\end{aligned}$$



Consequently,

$$P^{v',1} = \bigcup_{S \in \mathcal{A}^{v',1}} S = \bigcup_{S \in \mathcal{A}^{v,1}} S = P^{v,1}.$$

Then we have for all  $i \in P^{v',1}$  for any  $S \in \mathcal{A}^{v',1}$  for which  $i \in S$

$$\gamma_i^{v',1} = \chi_i^{v',1}(S) = a\chi_i^{v,1}(S) + b = a\gamma_i^{v,1} + b.$$

Let  $k \in \mathbb{N}$  and assume that  $\chi_i^{v',k}(S) = a\chi_i^{v,k}(S) + b$  for all  $S \in 2^N \setminus \{\emptyset\}$  and any  $i \in S$ ,  $\mathcal{A}^{v',k} = \mathcal{A}^{v,k}$ ,  $P^{v',k} = P^{v,k}$  and  $\gamma_i^{v',k} = a\gamma_i^{v,k} + b$  for all  $i \in P^{v',k}$ . Then we can write for all  $S \in 2^N \setminus \{\emptyset\}$  and any  $i \in S$

$$\begin{aligned} \chi_i^{v',k+1}(S) &= \begin{cases} \gamma_i^{v',k} & \text{if } i \in S \cap P^{v',k}; \\ \frac{v'(S) - \sum_{j \in S \cap P^{v',k}} \gamma_j^{v',k}}{|S \setminus P^{v',k}|} & \text{if } i \in S \setminus P^{v',k} \end{cases} \\ &= \begin{cases} a\gamma_i^{v,k} + b & \text{if } i \in S \cap P^{v,k}; \\ \frac{av(S) + b|S| - \sum_{j \in S \cap P^{v,k}} (a\gamma_j^{v,k} + b)}{|S \setminus P^{v,k}|} & \text{if } i \in S \setminus P^{v,k} \end{cases} \\ &= \begin{cases} a\gamma_i^{v,k} + b & \text{if } i \in S \cap P^{v,k}; \\ a \left( \frac{v(S) - \sum_{j \in S \cap P^{v,k}} \gamma_j^{v,k}}{|S \setminus P^{v,k}|} \right) + b & \text{if } i \in S \setminus P^{v,k} \end{cases} \\ &= a\chi_i^{v,k+1}(S) + b. \end{aligned}$$

Then we have

$$\begin{aligned} \mathcal{A}^{v',k+1} &= \{S \in 2^N \setminus \{\emptyset\} \mid \sum_{i \in S} \chi_i^{v',k+1}(S) = v'(S), \forall i \in S \forall T \in 2^N : i \in T : \chi_i^{v',k+1}(T) \leq \chi_i^{v',k+1}(S)\} \\ &= \{S \in 2^N \setminus \{\emptyset\} \mid \sum_{i \in S} (a\chi_i^{v,k+1}(S) + b) = av(S) + b|S|, \\ &\quad \forall i \in S \forall T \in 2^N : i \in T : a\chi_i^{v,k+1}(T) + b \leq a\chi_i^{v,k+1}(S) + b\} \\ &= \{S \in 2^N \setminus \{\emptyset\} \mid \sum_{i \in S} \chi_i^{v,k+1}(S) = v(S), \forall i \in S \forall T \in 2^N : i \in T : \chi_i^{v,k+1}(T) \leq \chi_i^{v,k+1}(S)\} \\ &= \mathcal{A}^{v,k+1}. \end{aligned}$$

Consequently,

$$P^{v',k+1} = \bigcup_{S \in \mathcal{A}^{v',k+1}} S = \bigcup_{S \in \mathcal{A}^{v,k+1}} S = P^{v,k+1}.$$

Then we have for all  $i \in P^{v',k+1}$  for any  $S \in \mathcal{A}^{v',k+1}$  for which  $i \in S$

$$\gamma_i^{v',k+1} = \chi_i^{v',k+1}(S) = a\chi_i^{v,k+1}(S) + b = a\gamma_i^{v,k+1} + b.$$

This means that for all  $k \in \mathbb{N}$  we have  $\chi_i^{v',k}(S) = a\chi_i^{v,k}(S) + b$  for all  $S \in 2^N \setminus \{\emptyset\}$  and any  $i \in S$ ,  $\mathcal{A}^{v',k} = \mathcal{A}^{v,k}$ ,  $P^{v',k} = P^{v,k}$  and  $\gamma_i^{v',k} = a\gamma_i^{v,k} + b$  for all  $i \in P^{v',k}$ . Then we have

$$n^{v'} = \min\{k \in \mathbb{N} \mid P^{v',k} = N\} = \min\{k \in \mathbb{N} \mid P^{v,k} = N\} = n^v,$$

$$\widehat{\mathcal{A}}^{v'} = \{S \in \mathcal{A}^{v',n^{v'}} \mid \forall_{T \in \mathcal{A}^{v',n^{v'}}} : S \not\subset T\} = \{S \in \mathcal{A}^{v,n^v} \mid \forall_{T \in \mathcal{A}^{v,n^v}} : S \not\subset T\} = \widehat{\mathcal{A}}^v$$

$$\text{and } D^{v'} = \bigcap_{S \in \widehat{\mathcal{A}}^{v'}} S = \bigcap_{S \in \widehat{\mathcal{A}}^v} S = D^v.$$

Moreover, for all  $i \in N$  we have

$$\widehat{\gamma}_i^{v'} = \gamma_i^{v', n^{v'}} = a\gamma_i^{v, n^v} + b = a\widehat{\gamma}_i^v + b.$$

This implies

$$\begin{aligned} & \alpha^{N \setminus D^{v'}, v'(N) - \sum_{j \in D^{v'}} \widehat{\gamma}_j^{v'}, (\widehat{\gamma}_j^{v'})_{j \in N \setminus D^{v'}}} \\ &= \min \left\{ t \in \mathbb{R} \left| \sum_{j \in N \setminus D^{v'}} \min\{\widehat{\gamma}_j^{v'}, t\} = v'(N) - \sum_{j \in D^{v'}} \widehat{\gamma}_j^{v'} \right. \right\} \\ &= \min \left\{ t \in \mathbb{R} \left| \sum_{j \in N \setminus D^v} \min\{a\widehat{\gamma}_j^v + b, t\} = av(N) + b|N| - \sum_{j \in D^v} (a\widehat{\gamma}_j^v + b) \right. \right\} \\ &= \min \left\{ t \in \mathbb{R} \left| \sum_{j \in N \setminus D^v} \min\{a\widehat{\gamma}_j^v, t - b\} + b|N \setminus D^v| = a \left( v(N) - \sum_{j \in D^v} \widehat{\gamma}_j^v \right) + b|N \setminus D^v| \right. \right\} \\ &= \min \left\{ t \in \mathbb{R} \left| \sum_{j \in N \setminus D^v} \min\{a\widehat{\gamma}_j^v, t - b\} = a \left( v(N) - \sum_{j \in D^v} \widehat{\gamma}_j^v \right) \right. \right\} \\ &= \min \left\{ t \in \mathbb{R} \left| \sum_{j \in N \setminus D^v} \min \left\{ \widehat{\gamma}_j^v, \frac{t - b}{a} \right\} = v(N) - \sum_{j \in D^v} \widehat{\gamma}_j^v \right. \right\} \\ &= a\alpha^{N \setminus D^v, v(N) - \sum_{j \in D^v} \widehat{\gamma}_j^v, (\widehat{\gamma}_j^v)_{j \in N \setminus D^v}} + b. \end{aligned}$$

Then we can write for all  $i \in N$

$$\begin{aligned} \Gamma_i(v') &= \begin{cases} \widehat{\gamma}_i^{v'} & \text{if } i \in D^{v'}; \\ \text{CEA}_i \left( N \setminus D^{v'}, v'(N) - \sum_{j \in D^{v'}} \widehat{\gamma}_j^{v'}, (\widehat{\gamma}_j^{v'})_{j \in N \setminus D^{v'}} \right) & \text{if } i \in N \setminus D^{v'} \end{cases} \\ &= \begin{cases} \widehat{\gamma}_i^{v'} & \text{if } i \in D^v; \\ \min \left\{ \widehat{\gamma}_i^{v'}, \alpha^{N \setminus D^{v'}, v'(N) - \sum_{j \in D^{v'}} \widehat{\gamma}_j^{v'}, (\widehat{\gamma}_j^{v'})_{j \in N \setminus D^{v'}}} \right\} & \text{if } i \in N \setminus D^v \end{cases} \\ &= \begin{cases} a\widehat{\gamma}_i^v + b & \text{if } i \in D^v; \\ \min \left\{ a\widehat{\gamma}_i^v + b, a\alpha^{N \setminus D^v, v(N) - \sum_{j \in D^v} \widehat{\gamma}_j^v, (\widehat{\gamma}_j^v)_{j \in N \setminus D^v}} + b \right\} & \text{if } i \in N \setminus D^v \end{cases} \\ &= \begin{cases} a\widehat{\gamma}_i^v + b & \text{if } i \in D^v; \\ a \min \left\{ \widehat{\gamma}_i^v, \alpha^{N \setminus D^v, v(N) - \sum_{j \in D^v} \widehat{\gamma}_j^v, (\widehat{\gamma}_j^v)_{j \in N \setminus D^v}} \right\} + b & \text{if } i \in N \setminus D^v \end{cases} \\ &= \begin{cases} a\widehat{\gamma}_i^v + b & \text{if } i \in D^v; \\ a\text{CEA}_i \left( N \setminus D^v, v(N) - \sum_{j \in D^v} \widehat{\gamma}_j^v, (\widehat{\gamma}_j^v)_{j \in N \setminus D^v} \right) + b & \text{if } i \in N \setminus D^v \end{cases} \\ &= a\Gamma_i(v) + b. \end{aligned}$$

Hence, the procedural egalitarian solution satisfies weak covariance.  $\square$

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